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1982 J. Phys. A: Math. Gen. 15 493

(http://iopscience.iop.org/0305-4470/15/2/020)

J. Phys. A: Math. Gen. 15 (1982) 493–506. Printed in Great Britain

Problems with metric–teleparallel theories of gravitation

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Received 20 March 1981

Abstract. A general Lagrangian formulation of the metric–affine, metric compatible theories of gravitation is given. The applicability of the metric–teleparallel geometry to gravitation is considered. It is pointed out that a teleparallel theory with the Lagrangian usually accepted in the literature leads to a non-predictable behaviour of torsion. A new choice of the Lagrangian is proposed.

1. Introduction

A theory of space–time based on the metric–teleparallel geometry was considered for the first time by Einstein (1928, 1929) in his attempt to create a unified theory of gravitation and electromagnetism. Nowadays theories of this kind have returned to physics following some different patterns of argumentation.

Hehl *et al* (1978) constructed a theory of gravitation (and, as they suggest, of strong interactions) based on the metric-torsion geometry following the scheme of the gauge theories. Their theory resembles the gauge theories used in microphysics, but has some distinct features. The linear connection is interpreted as the gauge potential corresponding to rotations, whereas the frames correspond to translations. The curvature and the torsion are analogous to the field strengths which correspond to rotations and translations respectively. The geometry is described by the metric tensor g and the linear connection ω or, equivalently, since the linear connection is metric compatible, by two tensor fields: the metric g and the torsion Q. A Lagrangian formulation of this class of theories is given in § 4.

In a similar way as in electrodynamics, where the electric current is the source of the electromagnetic field, in the theory of Hehl *et al* the spin of matter fields is the source of the curvature and their energy–momentum is the source of the torsion. In macrophysics the spin in most circumstances vanishes. Thus, as the authors suggest, the macrophysical curvature will vanish too. Therefore, in macrophysics we would have a metric compatible connection with vanishing curvature and non-vanishing torsion. The macrophysical theory of gravitation would be a theory based on the teleparallel geometry considered as a limit of the full microscopic theory of gravitation. The detailed exposition of this line of reasoning can be found in Hehl (1979) and Hehl *et al* (1980). A similar theory, but based on a different Lagrangian, was constructed by Wallner (1980).

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0305-4470/82/020493+14$02.00

Putting the spin equal to zero in the equations of Hehl *et al* does not force the curvature to vanish. Moreover, if we simultaneously let the spin and the curvature be zero in these equations, we get a contradiction, since then the torsion and in turn the energy–momentum tensor must also vanish. The contradiction can be avoided if in addition we turn a physical constant into zero. But, if this is the only way to derive the macrophysical field equations, I doubt the validity of the limiting hypothesis.

Hayashi and Shirafuji (1979) also took the point of view of the gauge theories, but, instead of the Poincaré group, they considered the translation group as the gauge group of gravitation. These authors, in contrast to Hehl *et al*, applied the teleparallel geometry to microphysics too. This is also the point of view of this paper.

Möller (1978) constructed a 'tetrad' theory of gravitation guided by the idea of averting the singularities inherent in the Einstein theory. As pointed out by Meyer (1981), his theory is formally the same as the teleparallel 'limit' of Hehl *et al.*

Schweizer and Straumann (1979), Nitsch and Hehl (1980) and Schweizer *et al* (1980) showed that from the observational point of view the teleparallel theory is practically indistinguishable from the Einstein theory.

In the present paper we conclude that in order to have a meaningful metric–teleparallel theory we should consider the interaction between gravity and spinning matter. The teleparallel field equations usually considered in the literature do not give full information about the teleparallel connection and, moreover, lead to inconsistencies in the non-vacuum case. Therefore, their modification is proposed.

The principal motivation for considering the metric–affine theories instead of the Einstein theory was to grant the spin independent dynamical meaning. This motivation is lost in the metric–teleparallel theories. On the other hand, the teleparallel geometry is simpler than the general affine geometry and this simplicity may be helpful in analysing such properties of the gravitational theory based on the Lagrangian (46) as renormalisability and the singularity problem.

2. The metric–affine geometry. Notation (Trautman 1972)

The metric–affine geometry is based on the metric field g and the linear connection ω.

We describe a geometric object as a law which with each local frame of 1-forms (θ^i) associates a collection (φ_A) of differential p-forms, $A, B, \ldots = 1, \ldots, N$. The type of a geometric object is characterised by a transformation law

$$(\varphi_A) \mapsto (\varphi'_A)$$

between the collections of p-forms corresponding to the frames (θ^i) and (θ'^i), where

$$\theta^i = a^i{}_j \theta'^j.$$

Let $\sigma\colon \mathrm{GL}(4, \mathbb{R}) \to \mathrm{GL}(N, \mathbb{R})$ be a representation of the group $\mathrm{GL}(4, \mathbb{R})$, $\sigma(a) = (\alpha^A{}_B(a))$, and $a = (a^i{}_j)$. If the transformation law is

$$\varphi'_A = \varphi_B \sigma^B{}_A(a) \tag{1}$$

then φ_A is called a p-form of type σ. Examples are the metric tensor g_{ij} ($p = 0$) and the frames θ^i themselves ($p = 1$)— in both cases σ is a tensorial representation.

The infinitesimal change of frames

$$\delta\theta^i = \theta'^i - \theta^i = -\varepsilon^i{}_j \theta^j \tag{2}$$

(or $a^i{}_j = \delta^i{}_j + \varepsilon^i{}_j$) induces the corresponding variation of the forms φ_A

$$\delta\varphi_A = -\varphi_B \sigma^{Bj}{}_{Ai} \varepsilon^i{}_j \tag{3}$$

where

$$\sigma^{Bj}{}_{Ai} = -\left.\frac{\partial \sigma^B{}_A(a)}{\partial a^i{}_j}\right|_{a=1}$$

provided φ_A as geometrical object is kept fixed.

The transformation law for the connection 1-forms $\omega^i{}_j$

$$a^i{}_k \omega'^k{}_j = \omega^i{}_k a^k{}_j + \mathrm{d}a^i{}_j \tag{4}$$

is different from the transformation law of p-forms of type σ (1). If φ_A is a p-form of type σ, then its covariant exterior derivative

$$D\varphi_A = \mathrm{d}\varphi_A + \sigma^{Bj}{}_{Ai}\omega^i{}_j \wedge \varphi_B \tag{5}$$

is a $(p+1)$-form of the same type σ.

Examples are the torsion 2-form

$$\Theta^i = D\theta^i = \mathrm{d}\theta^i + \omega^i{}_j \wedge \theta^j \tag{6}$$

and the covariant derivative of the metric tensor

$$Dg_{ij} = \mathrm{d}g_{ij} - \omega^k{}_i g_{kj} - \omega^k{}_j g_{ik}.$$

If φ_A is a 0-form, we write

$$D\varphi_A = \theta^i \nabla_i \varphi_A.$$

If we wish to extend the covariant differentiation to the Dirac field, we should impose the metric condition

$$Dg_{ij} = 0. \tag{7}$$

With this condition the geometry is based on two tensor fields: the metric g_{ij} and the torsion $Q^i{}_{jk}$ (determined by $\Theta^i = \frac{1}{2}Q^i{}_{jk}\theta^j \wedge \theta^k$), since the connection is given by

$$\omega^i{}_j = \tilde{\omega}^i{}_j - \tfrac{1}{2}(Q^i{}_{jk} + Q_{kj}{}^i + Q_{jk}{}^i)\theta^k \tag{8}$$

where $\tilde{\omega}^i{}_j$ is the Levi–Civita connection uniquely determined by the metric tensor. Due to the condition (7) the connection on the bundle of linear frames (with the structure group $\mathrm{GL}(4, \mathbb{R})$) can be reduced to the bundle of orthonormal frames (with the Lorentz group as a structure group). We shall assume the metric condition (7) throughout the paper.

The curvature 2-form $\Omega^i{}_j$ which appears in the Ricci identity

$$D^2\varphi_A = \sigma^{Bj}{}_{Ai}\Omega^i{}_j \wedge \varphi_B$$

is given by

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j = \tfrac{1}{2}R^i{}_{jkl}\theta^k \wedge \theta^l. \tag{9}$$

It will be useful to introduce the defect 1-form

$$\varkappa^i{}_j = \omega^i{}_j - \tilde{\omega}^i{}_j = \varkappa^i{}_{jk}\theta^k$$

which can be expressed in terms of the torsion by means of equation (8). The inverse relation is

$$\Theta^i = \varkappa^i{}_j \wedge \theta^j. \tag{10}$$

From equation (4) we obtain that the infinitesimal transformation of frames (2) leads to

$$\delta\omega^i{}_j = D\varepsilon^i{}_j. \tag{11}$$

3. The metric compatible teleparallel geometry

The metric compatible teleparallel geometry is a specialisation of the metric–torsion geometry for which the curvature vanishes, $\Omega^i{}_j = 0$. This requirement is, in a sense, opposite to the requirement of vanishing torsion, which is characteristic for the Riemannian geometry. The metric compatible teleparallel geometry, besides the description in terms of g_{ij} and $\omega^i{}_j$, can be described locally in a different manner.

Let $(\theta^i)_m$ be an orthonormal frame of the cotangent space at the point m of space–time. Let M be a simply connected neighbourhood of the point m. Then the parallel transport of $(\theta^i)_m$ along the curves lying entirely in M does not depend on the curves since the curvature vanishes and each two curves are homotopic. Thus $(\theta^i)_m$ generates the frame field $(\theta^i)_M$ on M. Starting with another orthonormal frame $(\theta'^i)_m$, we obtain the frame field $(\theta'^i)_M$ on M related to the previous one by a constant Lorentz transformation

$$\theta^i = \Lambda^i{}_j\theta'^j \qquad \Lambda^i{}_j = \text{constant}. \tag{12}$$

So, the geometry allows us to determine a class $[(\theta^i)_M]$ of OT (orthonormal teleparallel) frames on M related by the equivalence relation (12).

Inversely, such a class of frames determines the teleparallel metric compatible geometry on M. The metric tensor on M is then defined by

$$g = g_{ij}\theta^i \otimes \theta^j \qquad (g_{ij}) = \operatorname{diag}(+1, -1, -1, -1), \tag{13}$$

and does not depend on the choice of a member $(\theta^i)_M$ of the class $[(\theta^i)_M]$. The metric compatible flat connection is defined by the condition

$$\omega^i{}_j = 0$$

for each member of this class.

If we want to extend this description of the geometry to the whole manifold, we should cover it by simply connected open sets M_α, on each of them determine the class of the OT frames $[(\theta^i)_{M_\alpha}]$ subject to the condition (12), and make sure that this condition holds also for members of two different classes associated with M_α and M_β on their intersection.

Some authors (cf Robertson 1932) use another definition of the teleparallel geometry. That definition requires a path-independent parallel transport law between each two distant points. That requirement justifies the name 'teleparallel geometry'. The definition adapted here is more general: it is equivalent locally, but not globally, since the parallel transport is path dependent. In particular a global OT frame field may not exist.

To give an example, consider the cylinder with the metric

$$ds^2 = dz^2 + d\varphi^2$$

and the linear connection given by

$$\omega^{\varphi}{}_{z} = -\omega^{z}{}_{\varphi} = \alpha\,\mathrm{d}\varphi \qquad \alpha = \text{constant},$$

$$\omega^{\varphi}{}_{\varphi} = \omega^{z}{}_{z} = 0.$$

One can easily check that the connection is metric compatible and flat. The equations of the parallel transport around a horizontal circle for the vector $v = (v^{\varphi}, v^{z})$ are

$$\mathrm{d}v^{\varphi}/\mathrm{d}\varphi + \alpha v^{z} = 0 \qquad \mathrm{d}v^{z}/\mathrm{d}\varphi - \alpha v^{\varphi} = 0.$$

So, if α is not an integer, the vector v does not return to the original position after passing the circle. This shows that an OT frame cannot be constructed all over the cylinder.

4. Variational principle for the metric–torsion theories

4.1. *The gravitational Lagrangian*

The gravitational field in this class of theories is described by the metric tensor and the metric compatible connection, $Dg_{ij} = 0$. We shall use in this and the next sections exclusively the (local) orthonormal frames (θ^{i}). Then, the metric tensor has the form (13) and the connection 1-forms are skew symmetric, $\omega_{ij} = -\omega_{ji}$. The curvature 2-forms are skew symmetric too, $\Omega_{ij} = -\Omega_{ji}$. With this fixed choice of g_{ij}, the geometry is determined by θ^{i} and $\omega^{i}{}_{j}$. We assume that the Lagrangian 4-form K for the pure gravitational field is a function of θ^{i}, $\omega^{i}{}_{j}$, $\mathrm{d}\theta^{i}$, $\mathrm{d}\omega^{i}{}_{j}$. Then, due to the identities (6) and (9), we can write

$$K = K(\theta^{i}, \omega^{i}{}_{j}, \Theta^{i}, \Omega^{i}{}_{j}).$$

I define the (left) derivatives of K with respect to its arguments by the formula

$$\delta K = \delta\theta^{i} \wedge \frac{\partial K}{\partial\theta^{i}} + \tfrac{1}{2}\delta\omega^{i}{}_{j} \wedge \frac{\partial K}{\partial\omega^{i}{}_{j}} + \delta\Theta^{i} \wedge \frac{\partial K}{\partial\Theta^{i}} + \tfrac{1}{2}\delta\Omega^{i}{}_{j} \wedge \frac{\partial K}{\partial\Omega^{i}{}_{j}}. \tag{14}$$

The derivatives $\partial K/\partial\omega^{i}{}_{j}$ and $\partial K/\partial\Omega^{i}{}_{j}$ are skew symmetric by definition.

Transforming the formula (14), we obtain

$$\partial K = -\delta\theta^{i} \wedge e_{i} + \tfrac{1}{2}\delta\omega^{i}{}_{j} \wedge c_{i}^{j} + d\left(\delta\theta^{i} \wedge \frac{\partial K}{\partial\Theta^{i}} + \tfrac{1}{2}\delta\omega^{i}{}_{j} \wedge \frac{\partial K}{\partial\Omega^{i}{}_{j}}\right)$$

where

$$e_{i} = -\partial K/\partial\theta^{i} - D(\partial K/\partial\Theta^{i})$$

is the Einstein 3-form and

$$c_{i}^{j} = \partial K/\partial\omega^{i}{}_{j} + 2\theta^{j} \wedge \partial K/\partial\Theta^{i} + D(\partial K/\partial\Omega^{i}{}_{j})$$

is the Cartan 3-form.

The equations

$$e_{i} = 0, \tag{15}$$

$$c_{i}^{j} = 0, \tag{16}$$

are vacuum gravitational equations resulting from the variational principle $\delta\int K = 0$.

The Lagrangian 4-form K should depend only on geometry and not on the choice of an orthonormal frame (θ^i). So, if we perform a Lorentz rotation $\theta'^i = \Lambda^i{}_j\theta^j$ accompanied by the corresponding change (4) of $\omega^i{}_j$, the 4-form K should remain invariant, $K' = K$. The coefficients of an infinitesimal variation of frames determined by equation (2) are in this case skew symmetric, $\varepsilon_{ij} = -\varepsilon_{ji}$. By means of the formulae (3) and (11) we obtain

$$\delta K = \varepsilon^i{}_j(\theta^j \wedge e_i - \tfrac{1}{2}Dc_i^j) + \tfrac{1}{2}d(\varepsilon^i{}_j\,\partial K/\partial\omega^i{}_j) = 0.$$

Since $\varepsilon_{[ij]}$ and $d\varepsilon_{[ij]}$ have arbitrary values at each point, we obtain two identities

$$\partial K/\partial\omega^i{}_j = 0, \tag{17}$$

$$Dc_{ij} = \theta_j \wedge e_i - \theta_i \wedge e_j. \tag{18}$$

According to the identity (17) the Lagrangian 4-form K does not depend explicitly on $\omega^i{}_j$. The identity (18) shows that the vacuum field equations (15) and (16) are not independent. The covariant exterior differential of the left-hand side of equation (16) vanishes because of equation (15), even if equation (16) is not satisfied.

We require next that the Lagrangian K be invariant under diffeomorphisms, since otherwise K would be explicitly position dependent. If h is a (local) diffeomorphism of space–time, then (Schweizer 1979)

$$K(h^*\theta^i, h^*\Theta^i, h^*\Omega^i{}_j) = h^*K(\theta^i, \Theta^i, \Omega^i{}_j). \tag{19}$$

Consider now a local one-parameter group of diffeomorphisms $\{h_t\}$. Substituting h_t instead of h into formula (19) and differentiating with respect to t at $t = 0$, we obtain

$$\underset{Z}{\mathscr{L}}K = \underset{Z}{\mathscr{L}}\theta^i \wedge \frac{\partial K}{\partial\theta^i} + \underset{Z}{\mathscr{L}}\Theta^i \wedge \frac{\partial K}{\partial\Theta^i} + \frac{1}{2}\underset{Z}{\mathscr{L}}\Omega^i{}_j \wedge \frac{\partial K}{\partial\Omega^i{}_j} \tag{20}$$

for the vector field Z generated by $\{h_t\}$.

Using the formula

$$\underset{Z}{\mathscr{L}}\omega = \mathrm{d}(Z \lrcorner\, \omega) + Z \lrcorner\, \mathrm{d}\omega,$$

one can transform the identity (20) to the form

$$A + \mathrm{d}B = 0$$

where both forms A and B are linear in Z. Since Z and derivatives of its components are pointwise arbitrary, we conclude that A and B should vanish. As a result of a long but straightforward calculation, using the identity (18), we obtain the explicit expressions for A and B:

$$A = (Z \lrcorner\, \theta^i)De_i - (Z \lrcorner\, \Theta^i) \wedge e_i + \tfrac{1}{2}(Z \lrcorner\, \Omega^i{}_j) \wedge c_i^j = 0, \tag{21}$$

$$B = (Z \lrcorner\, \theta^i)\frac{\partial K}{\partial\theta^i} + (Z \lrcorner\, \Theta^i) \wedge \frac{\partial K}{\partial\Theta^i} + \tfrac{1}{2}(Z \lrcorner\, \Omega^i{}_j) \wedge \frac{\partial K}{\partial\Omega^i{}_j} - Z \lrcorner\, K = 0. \tag{22}$$

We have

$$Z \lrcorner\, \theta^i = Z^i, \qquad Z \lrcorner\, \Theta^i = Z^jQ^i{}_j, \qquad Z \lrcorner\, \Omega^i{}_j = Z^kR^i{}_{jk},$$

where the 1-forms $Q^i{}_j$ and $R^i{}_{jk}$ are defined by

$$Q^i{}_j = Q^i{}_{jk}\theta^k, \qquad R^i{}_{jk} = R^i{}_{jkl}\theta^l,$$

and let

$$Z \lrcorner K = Z^i K_i$$

be the definition of the 3-forms K_i. Then, we can use the fact that the identities (21) and (22) hold for every Z and rewrite them as

$$De_i = Q^j{}_i \wedge e_j - \tfrac{1}{2} R^j{}_{ki} \wedge c_j{}^k, \tag{23}$$

$$\frac{\partial K}{\partial \theta^i} + Q^j{}_i \wedge \frac{\partial K}{\partial \Theta^j} + \tfrac{1}{2} R^j{}_{ki} \wedge \frac{\partial K}{\partial \Omega^j{}_k} - K_i = 0. \tag{24}$$

The identity (23) corresponds to the contracted Bianchi identities of the Einstein theory, $\tilde{D}\tilde{e}_i = 0$. It shows that by covariant differentiation of the first field equation (15) we obtain an equation algebraically dependent on equations (15) and (16). The identity (24) can be transformed into an equivalent definition of the Einstein 3-forms,

$$e_i = Q^j{}_i \wedge \frac{\partial K}{\partial \Theta^j} + \tfrac{1}{2} R^j{}_{ki} \wedge \frac{\partial K}{\partial \Omega^j{}_k} - K_i - D \frac{\partial K}{\partial \Theta^i}.$$

4.2. The non-gravitational Lagrangian

The p-form of type σ (φ_A) will be the model of a field interacting with gravitation. We shall not specify whether φ_A is of bosonic or fermionic character. This 'unified' treatment is possible because of the metric condition, $Dg_{ij} = 0$.

The Lagrangian 4-form for the interaction will be

$$K + L$$

where L is the matter Lagrangian which depends on

$$\varphi_A,\ \mathrm{d}\varphi_A,\ \theta^i,\ \omega^i{}_j.$$

Using equation (5), we can also write that

$$L = L(\varphi_A, D\varphi_A, \theta^i, \omega^i{}_j).$$

Varying L, we obtain

$$\delta L = \delta\varphi_A \wedge L^A - \delta\theta^i \wedge t_i + \tfrac{1}{2}\delta\omega^i{}_j \wedge s_i{}^j + \mathrm{d}(\delta\varphi_A \wedge \partial L/\partial D\varphi_A)$$

where

$$L^A = \partial L/\partial\varphi_A - (-1)^p D(\partial L/\partial D\varphi_A),$$

$$t_i = -\frac{\partial L}{\partial \theta^i}, \qquad s_i{}^j = \frac{\partial L}{\partial \omega^i{}_j} + 2\sigma^{B[j}{}_{Ai]}\varphi_B \wedge \frac{\partial L}{\partial D\varphi_A}.$$

Thus the principle of least action, $\delta \int (K + L) = 0$, implies that

$$L^A = 0, \tag{25}$$

$$e_i = -t_i, \tag{26}$$

$$c_i{}^j = -s_i{}^j. \tag{27}$$

t_i and $s_i{}^j$ are 3-forms of material energy–momentum and spin respectively.

By the same method as for the gravitational Lagrangian K, we can investigate the consequences of the invariance of the matter Lagrangian L under the Lorentz rotations and diffeomorphisms. The invariance under the infinitesimal change of an orthonormal frame, $\delta\theta^i = -\varepsilon^i{}_j\theta^j$, leads to the equation

$$\delta L = \varepsilon^i{}_j(\theta^j \wedge t_i - \sigma^{Bj}{}_{Ai}\varphi_B \wedge L^A - \tfrac{1}{2}Ds_i{}^j) + \tfrac{1}{2}\mathrm{d}(\varepsilon^i{}_j\, \partial L/\partial\omega^i{}_j) = 0$$

from which we obtain two identities:

$$\partial L/\partial\omega^i{}_j = 0, \tag{28}$$

$$Ds_{ij} = \theta_j \wedge t_i - \theta_i \wedge t_j - 2\sigma^B{}_{A[ij]}\varphi_B \wedge L^A \tag{29}$$

Because of equation (28) the matter Lagrangian L does not depend explicitly on the connection. If the non-gravitational field equations are satisfied, then equation (29) becomes the covariant conservation law for angular momentum.

The invariance of the matter Lagrangian L under diffeomorphisms leads to the equation

$$\underset{Z}{\mathscr{L}}L = \underset{Z}{\mathscr{L}}\varphi_A \wedge \frac{\partial L}{\partial\varphi_A} + \underset{Z}{\mathscr{L}}D\varphi_A \wedge \frac{\partial L}{\partial D\varphi_A} - \underset{Z}{\mathscr{L}}\theta^i \wedge t_i. \tag{30}$$

Equation (30) can be transformed into the form

$$A + \mathrm{d}B = 0$$

where

$$A = (Z \lrcorner\, \theta^i)Dt_i - (Z \lrcorner\, \Theta^i) \wedge t_i + \tfrac{1}{2}(Z \lrcorner\, \Omega^i{}_j) \wedge s_i{}^j$$
$$+ (-1)^p(Z \lrcorner\, \varphi_A) \wedge DL^A + (Z \lrcorner\, D\varphi_A) \wedge L^A = 0 \tag{31}$$

and

$$B = (Z \lrcorner\, \varphi_A) \wedge \partial L/\partial\varphi_A + (Z \lrcorner\, D\varphi_A) \wedge \partial L/\partial D\varphi_A - (Z \lrcorner\, \theta^i) \wedge t_i - Z \lrcorner\, L = 0. \tag{32}$$

When φ_A is a 0-form, from equations (31) and (32) we obtain the identities

$$Dt_i = Q^j{}_i \wedge t_j - \tfrac{1}{2}R^j{}_{ki} \wedge s_j{}^k - \nabla_i\varphi_A L^A, \tag{33}$$

$$t_i = \nabla_i\varphi_A\, \partial L/\partial D\varphi_A - L_i, \tag{34}$$

where L_i is defined by $Z \lrcorner\, L = Z^i L_i$. If the non-gravitational field equations are satisfied, then equation (33) becomes the covariant conservation law of energy–momentum. It follows from the identities (18), (23), (29) and (33) that the equations resulting from covariant differentiation of the gravitational field equations (26) and (27) are algebraic consequences of the field equations themselves. The identity (34) means that the energy–momentum 3-form t_i is 'canonical'. Namely, defining the tensor t_{ji} by $t_i = {}^*\theta^j t_{ji}$, one obtains for t_{ji} the well known formula for the canonical energy–momentum tensor.

The electromagnetic field (both in the linear and nonlinear theory) is of special interest. The electromagnetic gauge invariance requires that the electromagnetic potential should be treated as a scalar 1-form $\varphi = \varphi_i\theta^i$ rather than as a covector 0-form. The Lagrangian L is a function of θ^i and of the field strength

$$F = \mathrm{d}\varphi = \tfrac{1}{2}F_{ij}\theta^i \wedge \theta^j.$$

In this case the spin 3-forms s_{ij} vanish and equation (32) leads to

$$t_i = F_i \wedge \partial L/\partial F - L_i$$

where $F_i = F_{ij}\theta^j$. This formula, though formally similar to formula (34), gives the symmetrical, gauge-invariant energy–momentum tensor t_{ji} rather than the canonical one.

The present derivation of the identities resulting from the invariance of the matter and gravitational Lagrangians is partially based on the results obtained independently by Schweizer (1979, 1980) and Adamowicz (1980). These identities in the framework of the Poincaré gauge gravitational theories were obtained also by means of other methods by Hehl *et al* (1978, 1980) and Szczyrba (1979).

5. Equations of the teleparallel theories

We attempt now to describe the gravitational field in the framework of the teleparallel metric compatible geometry. The gravitational Lagrangian will be a function of the orthonormal frame and the torsion, $K = K(\theta^i, \Theta^i)$. We cannot repeat directly the variational procedure of the previous section, since the vanishing of the curvature, $\Omega^i{}_j = 0$, imposes a constraint on the variation of the connection 1-forms $\omega^i{}_j$. To include this constraint in the variational formalism, we may choose one of the two methods described below.

The first is the method of Lagrange multipliers (cf Kopczyński 1975). The total Lagrangian $K + L$ is replaced here by the Lagrangian

$$K + L + \tfrac{1}{2}\lambda_i{}^j \wedge \Omega^i{}_j$$

where the 2-forms $\lambda_i{}^j$ ($\lambda_{ij} = -\lambda_{ji}$) are the Lagrange multipliers. From the corresponding principle of least action, by the variation with respect to φ_A, θ^i and $\omega^i{}_j$, we obtain the equations

$$L^A = 0, \tag{35}$$

$$e_i = -t_i, \tag{36}$$

$$c_i^{\;j} + s_i^{\;j} = -D\lambda_i^{\;j}, \tag{37}$$

respectively. The variation with respect to $\lambda_i{}^j$ gives the equation $\Omega^i{}_j = 0$.

It should be stressed that all identities derived in the previous section are valid also in the teleparallel case, provided we substitute $\Omega^i{}_j = 0$. By means of the identities (18) and (29) one can show that the integrability condition for equation (37)

$$D(c_i^{\;j} + s_i^{\;j}) = 0$$

is satisfied if equations (35) and (36) are satisfied. Therefore equation (37) can be locally solved for the Lagrange multipliers $\lambda_i{}^j$. Thus the only role of equation (37) is to determine (non-uniquely) the Lagrange multipliers.

As we know from § 3, the OT frame θ^i determines the teleparallel metric compatible geometry. Thus a more straightforward method of deriving the field equations is to use the OT frame as the unique variational variable corresponding to the gravitational field. For such frames the torsion coincides with the object of anholonomity, $\Theta^i = \mathrm{d}\theta^i$. The total Lagrangian $K + L$ depends on θ^i, φ_A, and their exterior derivatives. The principle of least action leads to equations (35) and (36) written in an OT frame. To write them in an arbitrary frame, we should replace the exterior derivative d by the covariant exterior derivative D. Both methods imply the same dynamical equations (35) and (36).

In contrast to the field equations in the metric–torsion theories (equations (25), (26) and (27)), in the teleparallel theories the spin is not explicitly present in the field equations (35) and (36). This is similar to the situation in the Einstein theory; however this analogy is not complete, since the canonical energy–momentum tensor t_{ji} rather than the symmetric tensor serves here as the source of gravitation (cf Rosenfeld 1940).

In order to have a meaningful metric–teleparallel theory, it is necessary to consider spinning matter. For, suppose we take into account the scalar p-form φ only. Such a scalar p-form can serve as a general model of non-spinning matter. Then the matter field equations (35) take the form

$$\partial L/\partial\varphi - (-1)^p \, \mathrm{d}(\partial L/\partial \, \mathrm{d}\varphi) = 0.$$

This equation generically depends on the metric tensor, but does not depend on the choice of a teleparallel connection. The same is true for the energy–momentum conservation law (33), which in the teleparallel space–time reduces to

$$Dt_i = Q^j{}_i \wedge t_j.$$

This equation can equivalently be written as

$$\tilde{D}t_i = (\varkappa^j{}_i + Q^j{}_i) \wedge t_j$$

or, using equation (10), as

$$\tilde{D}t_i = \varkappa^j{}_{ki}\theta^k \wedge t_j. \tag{38}$$

For non-spinning matter the energy–momentum tensor is symmetric, $\theta_k \wedge t_j = \theta_j \wedge t_k$; thus we obtain the equation

$$\tilde{D}t_i = 0$$

which is independent of the choice of the teleparallel connection. Considering non-spinning matter only, there is no other method of measuring the torsion than to subtract it from the field equations (36), substituting there the values of all other quantities measured in advance. Then the only role of these equations would be to provide the method to determine certain combinations of the torsion and its derivatives. As a physical law, equation (36) would in fact be meaningless.

On the other hand, considering spinning matter, we should be able to elaborate at least a thought experiment determining torsion. One can expect that such an experiment would be based on equation (38) or, as Adamowicz and Trautman (1975) suggest, on the conservation law of angular momentum (29). This law is also torsion dependent,

$$\tilde{D}s_{ij} - \theta_j \wedge t_i + \theta_i \wedge t_j = \varkappa^k{}_i \wedge s_{kj} + \varkappa^k{}_j \wedge s_{ik}.$$

This would solve the measurability problem posed by Möller (1978).

Invoking the analogy of the teleparallel theory with Maxwell's electrodynamics, we can restrict all possible Lagrangians to those quadratic in the torsion. There exist three independent Lagrangians with correct parity (Rumpf 1978):

$$K^1 = \tfrac{1}{2}(\theta^i \wedge \Theta^j) \wedge {}^*(\theta_j \wedge \Theta_i),$$

$$K^2 = \tfrac{1}{2}(\theta^i \wedge \Theta_i) \wedge {}^*(\theta^j \wedge \Theta_j),$$

$$K^3 = \tfrac{1}{2}\Theta^i \wedge {}^*\Theta_i.$$

Each Lagrangian K will be a linear combination of K^1, K^2 and K^3.

The Einstein 3-forms corresponding to these Lagrangians are, respectively,

$$e^1{}_i = D(^*(\theta_i \wedge \Theta_j) \wedge \theta^j) - \tfrac{1}{2}\Theta^j \wedge {}^*(\theta_j \wedge \Theta_i) - \tfrac{1}{2}{}^*(\theta^j \wedge \Theta^k) \wedge \theta_k \wedge Q_{ji} + \tfrac{1}{4}Q^{jkl}\eta_{klmi}\theta_j \wedge \Theta^m,$$

$$e^2{}_i = \theta_i \wedge D\,{}^*(\theta^j \wedge \Theta_j) - 2\Theta_i \wedge {}^*(\theta^j \wedge \Theta_j) + \tfrac{3}{4}{}^*(\theta^l \wedge \Theta_l) \wedge Q_{[ijk]}\theta^j \wedge \theta^k + \tfrac{1}{4}Q^{jkl}\eta_{jkli}\theta^m \wedge \Theta_m,$$

$$e^3{}_i = -D\,{}^*\Theta_i + \tfrac{1}{2}Q^j{}_i \wedge {}^*\Theta_j + \tfrac{1}{4}Q^{jkl}\eta_{klmi}\theta^m \wedge \Theta_j.$$

If we require that the weak field approximation gives the same results as those of the Einstein theory, we can restrict our considerations to the following one-parameter family of Lagrangians (Zaycoff 1929, Möller 1978, Nitsch 1979, Schweizer and Straumann 1979):

$$K = -K^1 + \tfrac{1}{2}\lambda K^2.$$

For the parameter value $\lambda = 1$ the Lagrangian K coincides, modulo an exact form, with the Einstein–Hilbert Lagrangian

$$\tilde{K} = \tfrac{1}{2}{}^*(\theta_i \wedge \theta^j) \wedge \tilde{\Omega}^i{}_j.$$

Thus, we have

$$K = \tilde{K} + \tfrac{1}{2}(\lambda - 1)K^2 + \text{an exact form}$$

and the gravitational field equations (36) take the form

$$\tilde{e}_i + \tfrac{1}{2}(\lambda - 1)e^2{}_i = -t_i \tag{39}$$

where

$$\tilde{e}_i = -e^1{}_i + \tfrac{1}{2}e^2{}_i$$

or, equivalently,

$$\tilde{e}_i = (\tilde{R}^j{}_i - \tfrac{1}{2}\tilde{R}\delta^j{}_i) \wedge {}^*\theta_j.$$

It is interesting to investigate invariance properties of the gravitational Lagrangian K. As in § 4.1, K is invariant under the Lorentz transformation of the orthonormal frames, accompanied by the corresponding passive transformation (4) of the connection 1-forms (which does not change the connection itself, but only its frame representation). However, the formulation of variational procedure which makes exclusive use of the OT frames suggests an investigation of the invariance properties of $K(\theta^i, \mathrm{d}\theta^i)$ under non-constant Lorentz transformations of the OT frames,

$$\theta^i \mapsto \theta'^i = \Lambda^i{}_j\theta^j \qquad \mathrm{d}\Lambda^i{}_j \neq 0. \tag{40}$$

In general, we have

$$K(\theta'^i, \mathrm{d}\theta'^i) \neq K(\theta^i, \mathrm{d}\theta^i).$$

In accordance with the results of § 4.1,

$$K(\theta^i, \mathrm{d}\theta^i) = K(\mathring{\theta}^i, D\mathring{\theta}^i)$$

where D is the covariant differential determined by the OT frame (θ^i) and $(\mathring{\theta}^i)$ is an arbitrary orthonormal frame. Thus the effect of the transformation (40) on K can be studied in terms of the resulting transformation of the connection and torsion forms, both taken with respect to a fixed frame $(\mathring{\theta}^i)$.

The 1-forms of the new connection with respect to the new OT frame (θ'') vanish, so the new connection 1-forms with respect to the old OT frame (θ^i) are

$$\omega'^i{}_j = \Lambda_k{}^i \, \mathrm{d}\Lambda^k{}_j.$$

Since the difference $\Delta\omega^i{}_j = \omega'^i{}_j - \omega^i{}_j$ is a tensorial 1-form, the formula

$$\Delta\omega^i{}_j = \Lambda_k{}^i D\Lambda^k{}_j$$

holds in an arbitrary frame $(\mathring{\theta}^i)$. Since the metric tensor does not change under the transformation (40), we have

$$\Delta\omega^i{}_j = \Delta\varkappa^i{}_j$$

and the increment of the torsion 2-form is

$$\Delta\Theta^i = \Delta\varkappa^i{}_j \wedge \mathring{\theta}^j.$$

Some authors (Kaempffer 1968, Cho 1976) require the gravitational Lagrangian $K(\theta^i, \mathrm{d}\theta^i)$ to be invariant (modulo an exact form) under the transformation (40) of OT frames. This means that only one of the ingredients of the geometry under consideration, namely the metric tensor, is present in the vacuum field equations. The second ingredient—the teleparallel connection—will not influence the left-hand side of the vacuum field equations. Therefore the theory which satisfies this requirement, i.e. the theory with $\lambda = 1$, cannot be interpreted as a metric–teleparallel theory.

The non-vacuum field equations (39) could lead in this case to an inconsistency if the matter Lagrangian is connection dependent, as in the case of the Dirac field. This inconsistency can be avoided by introducing into the matter Lagrangian L *ad hoc* the Levi–Civita connection $\tilde{\omega}^i{}_j$ instead of the dynamical connection $\omega^i{}_j$. From the point of view of the formalism considered here it is, however, not allowed since in such a case L would be explicitly $\mathrm{d}\theta^i$ (torsion)-dependent and the principle of minimal coupling would be violated (cf Thirring 1980). The opposite point of view is that of Cho (1978), who admits such a violation of this principle in the gravitational gauge theories.

For $\lambda \neq 1$ the Lagrangian K is not invariant under all transformations (40); there exist however non-constant Lorentz transformations such that ${}_A\Theta = \mathring{\theta}^i \wedge \Theta_i$ is left invariant. These transformations do not change $\tilde{K}$ and, since the duality operation is determined by the metric tensor only, K^2 does not change either. The 3-form ${}_A\Theta$ (determined by the axial part of the torsion) transforms according to the formula

$${}_A\Theta \mapsto {}_A\Theta - \Delta\varkappa_{ij} \wedge \mathring{\theta}^i \wedge \mathring{\theta}^j;$$

thus the condition of invariance of ${}_A\Theta$ reads

$$\Delta_A\Theta \equiv -\Delta\varkappa_{ij} \wedge \mathring{\theta}^i \wedge \mathring{\theta}^j = 0. \tag{41}$$

Although the Lagrangian K^2 is invariant under the transformations (40) which satisfy the condition (41), the corresponding Euler–Lagrange equations, in general, are not. The reason is that in the variational procedure equation (41) plays a role of a constraint imposed on the variational variables θ^i. Under (40) K^2 transforms like

$$K^2 \mapsto K^2 + \Delta_A\Theta \wedge {}^*_A\Theta + \tfrac{1}{2}\Delta_A\Theta \wedge {}^*\Delta_A\Theta. \tag{42}$$

First performing the variation of the expression (42) and next using the condition (41), we obtain

$$e^2{}_i \mapsto e^2{}_i - 2\Delta\varkappa_{ij} \wedge \mathring{\theta}^j \wedge {}^*_A\Theta.$$

Therefore, sufficient conditions for the invariance of the empty space field equations (39) under the transformations (40) are equation (41) and the equation

$$\Delta\varkappa_{ij} \wedge \overset{\circ}{\theta}{}^{j} \wedge {}^{*}_{A}\Theta = 0. \tag{43}$$

The set of solutions of equations (41) and (43) is non-empty; thus the vacuum field equations (39) (together with initial data) are insufficient to determine the torsion tensor completely. To illustrate this I shall construct an example. Another example of this kind can be found in Baekler (1980).

Consider the Minkowski space

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{44}$$

equipped with the common Levi–Civita connection, $\tilde{\omega}^{i}{}_{j} = 0$. This is of course a solution of the vacuum field equations. Consider however in the Minkowski space another metric compatible teleparallel connection

$$\omega^{i}{}_{j} = K^{i}{}_{j}(t)\, dt$$

where $K_{ij} = -K_{ji}$ and $K_{ab} = 0$ for $a, b = 1, 2, 3$. Since then the axial part of the torsion vanishes, ${}_{A}\Theta = 0$, we have another solution of the vacuum field equations (39). The functions $K^{0}{}_{a}$, which determine the torsion field,

$$Q^{0}{}_{0a}(t) = K^{0}{}_{a}(t), \tag{45}$$

are arbitrary functions of the coordinate t. We see that from the knowledge of the torsion on the initial hypersurface $t = 0$ we cannot predict the behaviour of some torsion components outside this hypersurface. This unpredictability of the torsion behaviour has apparently a geometric character in contrast to the unpredictability of some metric components in the Einstein theory.

The torsion, if it exists, should be a measurable physical quantity, like the metric. Therefore, the teleparallel theory described above is principally incorrect. The opposite point of view can, however, be found in the literature. Hayashi and Shirafuji (1979) recognise certain mutually different teleparallel geometries as physically equivalent. Then, for instance, the solution given by equations (44) and (45) would be interpreted as a 'pure gauge'. The Dirac Lagrangian and the Dirac equation are indeed invariant under the transformations (40) satisfying the condition (41), since they depend only on the axial part of the torsion tensor. The energy–momentum tensor of the Dirac field is, however, not invariant under these transformations. Equations (38) and (39) are then not invariant under these transformations, which shows that the interpretation of Hayashi and Shirafuji is internally inconsistent. Also the Rarita–Schwinger equation has not the invariance property considered here.

To rescue the metric–teleparallel theory one can attempt to modify the field equations (39). For instance, one can consider non-quadratic Lagrangians, as was suggested by Möller (1978). The simplest way is, however, to consider an admixture of the 4-form K^3 to the gravitational Lagrangian,

$$K = \lambda K^1 + \mu K^2 + \nu K^3. \tag{46}$$

The Lagrangian (46) has in fact two free parameters, since a combination of the parameters λ and ν is the gravitational constant (which is chosen in this paper to be $1/8\pi$). For $\nu \neq 0$ there is no additional invariance of the Lagrangian and the field equations will, we hope, have the required evolutional character. If ν is sufficiently small, then the measurable results of the theory ought to be close to the Einsteinian values, which would ensure the compatibility of the theory with astronomical observations. The analysis of Hayashi and Shirafuji (1979) gives that ν can be at most of the order 10^{-3}. It is not clear, however, whether the Lagrangian (46) does not lead to the dipole radiation catastrophe.

Acknowledgment

I am indebted to Professor Friedrich W Hehl for numerous discussions.

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